International Journal of Theoretical Physics, Vol. 31, No. 9, 1992

Completeness in Sums of Boolean Algebras and Logics

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Received February 10, 1992

1. INTRODUCTION

The basis of this work is the sum of a Boolean algebra and a logic (orthomodular lattice) defined in Pták (1986). For a given Boolean algebra B and a logic L_1 it enables us to construct a logic L such that both B and L_1 are sublogics of L and many properties of B and L_1 are inherited for L (e.g., if L_1 is a modular logic, then L is also modular), while a horizontal sum ($\{0; 1\}$-pasting) of B and L_1 does not respect them. These questions are studied in Janiš (1990). On the other hand, completeness of both B and L_1 does not guarantee completeness of L. This work deals with some aspects of MacNeille completions of sums and with relations between sums and direct products of logics.

2. BASIC NOTIONS AND DEFINITIONS

By an *orthomodular lattice* (OML) we understand a lattice L with 0, 1 and with an orthocomplementation (we denote by a' the orthocomplement of an element a) fulfilling the following conditions:

(i) $(a')' = a$ for each $a \in L$.
(ii) If $a \leq b$, then $b' \leq a'$ for each $a, b \in L$.
(iii) $a \vee a' = 1$ for each $a \in L$.
(iv) If $a, b \in L$, $a \leq b$, then $b = a \vee (b \wedge a')$ (the orthomodular law).

The elements a, b are *orthogonal* if $a \leq b'$. The term *logic* (quantum logic) is used for orthomodular lattices as well.

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0020-7748/92/0900-1689$06.50/0

If L_1, L_2 are OMLs, then the mapping $f: L_1 \to L_2$ is called a *morphism* if for each a, $b \in L_1$

$$f(0)=0, \qquad f(a \vee b)=f(a) \vee f(b) \qquad \text{and} \qquad f(a')=(f(a))'$$

If f is bijective and both f and f^{-1} are morphisms, then f is called an *isomorphism.* The mapping $f: L_1 \to L_2$ is called an *embedding* if $f: L_1 \to f(L_1)$ is an isomorphism.

We introduce the definition of a *sum of a Boolean algebra and a logic* as it is used in Pták (1986).

If B is a Boolean algebra and L_1 is an OML, then the OML L is called a sum of B and L_1 if the following conditions are fulfilled:

(i) There exist embeddings $f: B \to L, f_1: L_1 \to L$ such that $f(a) \wedge f_1(b) = 0$ if and only if $a=0$ or $b=0$; $a \in B$, $b \in L_1$.

(ii) There is no proper sublogic of L containing $f(B) \cup f_1(L_1)$.

(iii) If s_1 and s_2 are states on B and L_1, respectively, then there is a state s on L such that $s(f(a))=s_1(a)$ for each $a \in B$ and $s(f_1(b))=s_2(b)$ for each $b \in L_1$.

It is proved in Pták (1986) that this sum exists for any Boolean algebra and any OML. Now we shall briefly describe its structure:

Let us consider the set S of all the elements of the form $\langle(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\rangle$, where n is a natural number, $a_i \in B$, and $b_i \in L_1$, $i = 1, 2, \ldots, n$, such that $a_i \wedge a_j = 0$ whenever $i \neq j$ and $\bigvee_{i=1}^{n} a_i = 1$.

The relation $\leq$ in S is defined in the following way: $\langle(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\rangle \leq \langle(c_1, d_1), (c_2, d_2), \ldots, (c_m, d_m)\rangle$ if $b_i \leq d_j$ whenever $a_i \wedge c_j \neq 0$.

If we identify those elements p, $r \in S$ for which both $p \leq r$ and $r \leq p$, then the set of all the corresponding equivalence classes is the required sum $B \oplus L_1$. We shall write its elements in square brackets.

The orthocomplementation in $B \oplus L_1$ is defined in the following way: If $p \in B \oplus L_1$,

$$p=[(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)]$$

then

$$p'=[(a_1, b_1'), (a_2, b_2'), \ldots, (a_n, b_n')]$$

It is easy to verify that if

$$p=[(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)]$$
$$r=[(c_1, d_1), (c_2, d_2), \ldots, (c_m, d_m)]$$

then

$$p \vee r = [(a_i \wedge c_j, b_i \vee d_j)]_{i=1,j=1}^{n,m}$$

$$p \wedge r = [(a_i \wedge c_j, b_i \wedge d_j)]_{i=1,j=1}^{n,m}$$

The embeddings f, f_1 from the definition of the sum are such that

$$f(a) = [(a, 1), (a', 0)], \qquad f_1(b) = [(1, b)] \qquad \text{for} \quad a \in B, \quad b \in L_1$$

3. PROPERTIES OF SUMS

In this section we give a brief review of some results proved in Pták (1986) and Janiš (1990) and derive two properties useful for our purposes. Throughout the whole section we suppose that B is a Boolean algebra and L_1 is an OML.

If K is an OML, then we denote its center by $C(K)$, i.e.,

$$C(K) = \{x \in K; x = (x \wedge y) \vee (x \wedge y') \text{ for every } y \in K\}$$

Proposition 3.1. $C(B \oplus L_1) = B \oplus C(L_1)$ (Pták, 1986, Theorem 2.1).

It follows from this proposition that the sum of two Boolean algebras is a Boolean algebra. (Note that the center of the $\{0; 1\}$-pasting is the set $\{0; 1\}$.)

Proposition 3.2. $B \oplus L_1$ is modular if and only if L_1 is modular (Janiš, 1990, Theorem 3.1).

Proposition 3.3. $B \oplus L_1$ is atomic if and only if both B and L_1 are atomic. The set of all atoms of the sum consists of the elements $[(a, b), (a', 0)]$, where a is an atom in B and b is an atom in L_1 (Janiš, 1990, Theorem 3.3).

The next proposition shows how the elements of the sum can be obtained by a finite number of lattice operations from elements of $f(B)$ and $f_1(L_1)$.

Proposition 3.4. For each $x \in B \oplus L_1$ there exist $n \in N$; $a_1, \ldots, a_n \in B$; $b_1, \ldots, b_n \in L_1$ such that

$$x = \bigvee_{i=1}^{n} (f(a_i) \wedge f_1(b_i))$$

Proof. Let $x = [(a_1, b_1), \ldots, (a_n, b_n)]$. Applying the proposed lattice operations on $f(a_1), \ldots, f(a_n)$ and $f_1(b_1), \ldots, f_1(b_n)$, where f and f_1 are the embeddings from the definition of the sum, we obtain the required representation of x. ■

The following statement deals with the strong almost orthogonality which was introduced in Pulmannová and Riečanová (1990).

Definition 3.5. Let L be an OML, $A \subset L$, A is a set of atoms. Then A is *strongly almost orthogonal* if for every $a \in A$ there is $n_a \in N$ such that each sequence $\{a_i\}_{i=1}^{\infty}$, $a_i \in A$, where $a_1 = a$, a_i is not orthogonal to a_{i+1}, $i = 1, 2, \ldots,$ contains at most n_a distinct elements.

Proposition 3.6. If the set of all atoms in L_1 is strongly almost orthogonal, then the set of all atoms in $B \oplus L_1$ is also strongly almost orthogonal.

Proof. If $p = [(a_1, b_1), (a_1', 0)]$, $q = [(a_2, b_2), (a_2', 0)]$ are atoms in $B \oplus L_1$ such that $a_1 \neq a_2$, then p is orthogonal to q. If $a_1 = a_2$, then p is orthogonal to q if and only if b_1 is orthogonal to b_2. Therefore, if $\{p_i\}_{i=1}^{\infty}$ is a sequence of atoms in $B \oplus L_1$ such that p_i is not orthogonal to p_{i+1}, $p_i = [(a_i, b_i), (a_i', 0)]$, then $\{b_i\}_{i=1}^{\infty}$ is a sequence of atoms in L_1 such that b_i is not orthogonal to b_{i+1} $(i = 1, 2, \ldots)$ and the proposition is proved. ■

4. COMPLETION OF THE SUM

This section contains the main results concerning the MacNeille completion of sums of Boolean algebras and OMLs. As we have already mentioned, completeness of B and L_1 may not be inherited for L. The next example [introduced also in Janiš (1990)] shows that even in case of both B and L_1 being complete Boolean algebras, their sum L may not be complete.

Example 4.1. Let B be the system of all subsets of the interval $[0; \infty)$, B is ordered by the natural set inclusion, and let $L_1 = B$. Then both B and L_1 are complete Boolean algebras. Denote by a_m^n the interval $[m; n)$, m, $n \in [0; \infty]$. Let us consider the set

$$M = \{[(a_0^1, a_0^1), (a_1^2, a_1^2), \ldots, (a_n^{n+1}, a_n^{n+1}), (a_{n+1}^{\infty}, a_{n+1}^{\infty})]\}_{n=0}^{\infty}$$

Then M is a subset of L and an easy computation shows that $\bigwedge M$ does not exist in L. Therefore L is not complete.

If K is a lattice, then by $\bar{K}$ we denote its *MacNeille completion*, i.e., $\bar{K}$ is a complete lattice in which K can be embedded by an embedding ϕ and every element of $\bar{K}$ is a join and a meet of elements of $\phi(K)$. [We say that $\phi(K)$ is join dense and meet dense in $\bar{K}$.] It is known that MacNeille completion of an OML need not be an OML (Kalmbach, 1983, p. 259). A sufficient condition for $\bar{K}$ to be an OML is, e.g., K being an atomic (o)-continuous OML such that the interval topology in K is Hausdorff (Riečanová, 1990). Another similar condition requires K to be an atomic OML with a strongly almost orthogonal set of all atoms (Pulmannová and Riečanová, 1990, Theorem 2.5).

The following theorem explains the relationship between summing of completions and completions of sums.

Theorem 4.2. Let B be a Boolean algebra, and let L_1 be an OML such that $\bar{L}_1$ is an OML. Then

$$\overline{B\oplus\bar{L}_1}=\overline{B\oplus L_1}$$

Proof. We prove the theorem in three steps:

(i) First we show $\bar{L}_1\subset\overline{B\oplus L_1}$ (i.e., there exists an embedding of $\bar{L}_1$ into $\overline{B\oplus L_1}$):

If $x\in\bar{L}_1$, then there exists a set $E\subset L_1$ such that $x=\bigvee E$. Let $f_1^*:\bar{L}_1\to\overline{B\oplus L_1}$ be such that for every $x\in L_1$

$$f_1^*(x)=\bigvee_{e\in E} f_1(e), \qquad \text{where} \quad x=\bigvee E$$

It is easy to show that f_1^* is an embedding.

(ii) We will show that $B\oplus\bar{L}_1\subset\overline{B\oplus L_1}$ (again in the sense of an embedding):

Denote by $\bar{f}_1$ the usual embedding $\bar{f}_1:\bar{L}_1\to B\oplus\bar{L}_1$, i.e., $\bar{f}_1(b)=[(1,b)]$ for each $b\in\bar{L}_1$.

Let $f_2:B\oplus\bar{L}_1\to\overline{B\oplus L_1}$ be defined in the following way: If $x\in B\oplus\bar{L}_1$, then (Proposition 3.4)

$$x=\bigvee_{i=1}^{n} (f(a_i)\wedge\bar{f}_1(b_i)), \qquad a_i\in B, \quad b_i\in\bar{L}_1, \quad i=1,2,\ldots,n$$

Put $f_2(x)=\bigvee_{i=1}^{n} (f(a_i)\wedge f_1^*(b_i))$. Then f_2 is the required embedding.

(iii) We have

$$\overline{B\oplus\bar{L}_1}=\overline{B\oplus L_1} \qquad \text{(in the sense of isomorphism)}$$

Indeed, as $B\oplus L_1\subset B\oplus\bar{L}_1$, the inclusion

$$\overline{B\oplus L_1}\subset\overline{B\oplus\bar{L}_1}$$

is obvious. The inverse follows from (ii). ■

If K is an atomic lattice, then obviously K and $\bar{K}$ have isomorphic sets of all atoms. Therefore, if B and L_1 are atomic, then the sets of all atoms of $B\oplus L_1$, $B\oplus\bar{L}_1$, and $\overline{B\oplus L_1}$ are isomorphic.

Theorem 4.3. If B is an atomic Boolean algebra, L_1 is an atomic OML, and the set of all atoms of L_1 is strongly almost orthogonal, then

$$\overline{B\oplus L_1}=\prod\,[0;f(a)\wedge f_1(b)]$$

where the product is over all pairs (a,b) such that a is an atom in B and b is an atom of $C(L_1)$. Moreover, $f(a)\wedge f_1(b)$ are atoms in $C(B\oplus L_1)$ and $[0;f(a)\wedge f_1(b)]$ are finite OMLs.

Proof. According to Proposition 3.6, the set of all atoms of $(B \oplus L_1)$ is strongly almost orthogonal. Hence by Pulmannová and Riečanová (1990), Theorem 2.3(iii), we obtain that $B \oplus L_1$ can be embedded into a direct product $\overline{B \oplus L_1} = \prod_{i \in I} [0; c_i]$, where c_i are atoms of $C(B \oplus L_1) = B \oplus C(L_1)$ and $[0; c_i]$ are finite OMLs. Moreover, the sets of all atoms of $B \oplus L_1$ and $\overline{B \oplus L_1}$ are isomorphic. It follows that $c_i = f(a_i) \wedge f_1(b_i)$, where a_i is an atom in B and b_i is an atom in $C(L_1)$, $i \in I$. ■

5. SUMS AND DIRECT PRODUCTS

The aim of this section is to show that in the case of a finite atomic Boolean algebra B and an arbitrary OML L_1 the sum $B \oplus L_1$ is isomorphic to the direct product of OMLs L_1 and in case of any atomic Boolean algebra B the sum is dense in such a product, which enables us to introduce a necessary and sufficient condition for completeness of the sum.

Theorem 5.1. Let B be an atomic Boolean algebra with n atoms and let L be an OML. Then the sum $B \oplus L_1$ is isomorphic to the direct product $L_1 \times L_1 \times \cdots \times L_1$ (n times).

Proof. Let $\{a_1, a_2, \ldots, a_n\}$ be the set of all atoms in B. Then each element of $B \oplus L_1$ can be written in the form

$$x = [(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)]$$

It is easy to prove that

$$f\colon \quad B \oplus L_1 \to L_1 \times L_1 \times \cdots \times L_1 \quad (n \text{ times})$$

$$f(x) = (b_1, b_2, \ldots, b_n)$$

is the required isomorphism. ■

This relationship is a little different if B is infinite:

Theorem 5.2. Let B be an atomic Boolean algebra and let L_1 be an OML. Then $B \oplus L_1$ is isomorphic to a join dense and meet dense subset of the direct product $\prod_{a \in A} L_a$, $L_a = L_1$ for each $a \in A$, where A is the set of all atoms in B.

Proof. The set $\prod_{a \in A} L_a$ consists of all the functions $f: A \to L_1$. Let $h\colon L \to \prod_{a \in A} L_a$ be defined in the following way:

If $x = [(c_1, d_1), (c_2, d_2), \ldots, (c_n, d_n)] \in L$, then for any $a \in A$ there exists exactly one $k(a) \in \{1; 2; \ldots; n\}$ such that $a \leq c_{k(a)}$. Then we put $h(x) = f_x$, where $f_x(a) = c_{k(a)}$. A routine verification shows that h is an embedding.

Now let $f \in \prod_{a \in A} L_a$, and let

$$M = \{x_a \in L;\, x_a = [(a, f(a)), (a', 0)],\, a \in A\}$$

Then for each $x_a \in M$ there is $h(x_a) \leq f$; therefore

$$f \geq \bigvee_{a \in A} h(x_a)$$

If $g \in \prod_{a \in A} L_a$, $g \geq \bigvee_{a \in A} h(x_a)$, then for each $a \in A$ we have $g(a) \geq f(a)$, which means $g \geq f$. We have proved that each element of $\prod_{a \in A} L_a$ is a join (and hence also a meet) of some elements of $h(L)$ and therefore $h(L)$ is join dense and meet dense in $\prod_{a \in A} L_a$. ■

Corollary 5.3. If B is an atomic Boolean algebra and L_1 is a complete OML, then the MacNeille completion of $B \oplus L_1$ is

$$\overline{B \oplus L_1} = \prod_{a \in A} L_a, \qquad L_a = L_1 \text{ for each } a \in A$$

where A is the set of all atoms in B.

Corollary 5.4. If B is an atomic Boolean algebra and L_1 is an OML such that its MacNeille completion $\bar{L}_1$ is an OML, then the MacNeille completion of $B \oplus L_1$ is

$$\overline{B \oplus L_1} = \prod_{a \in A} L_a, \qquad L_a = \bar{L}_1 \quad \text{for each } a \in A$$

where A is the set of all atoms in B.

Proof. In view of Theorem 4.2 and Corollary 5.3, we have

$$\overline{B \oplus L_1} = \overline{B \oplus \bar{L}_1} = \prod_{a \in A} L_a, \qquad L_a = \bar{L}_1 \quad \text{for each } a \in A \quad ■$$

Let us now consider another special case when the OML L_1 is finite.

Theorem 5.5. Let B be an atomic and complete Boolean algebra and let L_1 be a finite OML. Let $P = \{(a, b) \mid a \text{ is an atom in } B,\, b \text{ is an atom in } C(L_1)\}$. Then $B \oplus L_1$ is isomorphic to

$$\prod_{(a,b) \in P} [0; f(a) \wedge f_1(b)]$$

where f, f_1 are the embeddings from the definition of $B \oplus L_1$ and $[0; f(a) \wedge f_1(b)]$ are finite OMLs.

Proof. In view of Theorem 4.3, we have

$$B \oplus L_1 \subset \overline{B \oplus L_1} = \prod_{(a,b) \in P} [0, f(a) \wedge f_1(b)]$$

(in the sense of an embedding).

Let $\{b_1, b_2, \ldots, b_n\}$ be the set of all atoms in L_1. Since the sets of all atoms of $B \oplus L_1$ and $\overline{B \oplus L_1}$ are isomorphic, for any $x \in \overline{B \oplus L_1}$ we have

$$x = \bigvee \{c \mid c \leq x,\ c \text{ is an atom in } \overline{B \oplus L_1}\} = \bigvee_{k=1}^{n} d_k$$

where

$$d_k = \bigvee \{c \mid c \leq x \text{ and there exists an atom } a \in B \text{ such that } c = f(a) \wedge f_1(b_k)\},$$

$$k = 1, 2, \ldots, n$$

Since B is complete and $f(a), f_1(b)$ are compatible, i.e.,

$$f(a) = ((f(a) \wedge f_1(b)) \vee (f(a) \wedge f_1(b)')) \quad \text{for any } a \in B,\ b \in L_1$$

we have

$$d_k = (\bigvee \{f(a) \mid f(a) \wedge f_1(b_k) \leq x,\ a \text{ is an atom in } B\}) \wedge f_1(b_k)$$

and thus $d_k \in B \oplus L_1$ and $x \in B \oplus L_1$ (in the sense of an isomorphism). ■

Due to Theorems 5.1 and 5.5, a sufficient condition for the completeness of $B \oplus L_1$ is that one of B, L_1 is finite and the other is complete (assuming both B and L_1 atomic). The next theorem shows that this condition is necessary, too. The idea of its proof is similar to that used in Example 4.1.

Theorem 5.6. If a Boolean algebra B and an OML L_1 are both infinite, then $B \oplus L_1$ is not complete.

Proof. Let $\{a_1, a_2, \ldots, a_n, \ldots\}$ be an infinite, countable, pairwise orthogonal set of nonzero elements in B and let $\{b_1, b_2, \ldots, b_n, \ldots\}$ be an infinite countable set of mutually different elements of L_1. Put

$$p_1 = [(a_1, b_1), (a_1', 1)]$$

$$p_2 = [(a_1, b_1), (a_2, b_2), (a_1' \wedge a_2', 1)]$$

$$\vdots$$

$$p_n = [(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n), (a_1' \wedge a_2' \wedge \cdots \wedge a_n', 1)]$$

$$\vdots$$

Then

$$p_i \in B \oplus L_1, \qquad i = 1, 2, \ldots, n, \ldots, \quad p_1 \geq p_2 \geq \cdots \geq p_n \geq \cdots$$

We will show that $\bigwedge_{i=1}^{\infty} p_i$ does not exist in $B \oplus L_1$: Suppose the contrary. Then

$$p_0 = \bigwedge_{i=1}^{\infty} p_i = [(c_1, d_1), (c_2, d_2), \ldots, (c_m, d_m)] \in B \oplus L_1$$

Evidently every a_n has nonvoid meet with at least one c_i, $i \in \{1, 2, \ldots, m\}$. We claim that then each c_i has nonvoid meet with at most one a_n, $i = 1, 2, \ldots, m$, $n = 1, 2, \ldots$, which contradicts the infinity of the set $\{a_1, a_2, \ldots\}$. Indeed, assume without loss of generality that $c_1 \wedge a_1 \neq 0$, $c_1 \wedge a_2 \neq 0$; then $d_1 \leq b_1 \wedge b_2$. Thus the element

$$[(c_1 \wedge a_1, b_1), (c_1 \wedge a_2, b_2), (c_1 \wedge a_1' \wedge a_2', d_1), (c_2, d_2), \ldots, (c_m, d_m)]$$

is greater than p_0 (since $b_1 \neq b_2$) and it is less than or equal to any of the elements p_i, $i = 1, 2, \ldots$, which is a contradiction. Hence $B \oplus L_1$ is not complete. ■

Now we have the following result:

Corollary 5.7. Let B be an atomic Boolean algebra and let L_1 be an atomic OML. Then $B \oplus L_1$ is complete if and only if one of them is complete and the other is finite.

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